Malliavin calculus of canonical stochastic differential equations with jumps

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Theory of the stochastic differential equation (SDE) based on a Brownian motion or a continuous semimartingale is now well developed. It provides a fundamental tool not only for the reseach of the stochastic analysis but of the stochastic control, filtering theory and mathematical finance. On the other hand, stochastic differential equations with jumps based on Lévy processes are not yet well understood. Many different types of the stochastic differential equations are studied, owing partly to the variety of the Lévy processes.

In this paper, we will discuss a *canonical SDE* with jumps. Among many SDE's with jumps, the canonical one has some nice geometric and analytic properties. In the next section, we give the defintion of the equation and state basic properties of the solution. The main part of this paper is in Sections 2-4, where we discuss the existence and the smoothness for the density function of the distribution of the solution to the canonical SDE.

1. Canonical SDE.

Canonical stochastic differential equation with jumps was first introduced by Marcus [11]. The solution of the equation has some nice geometric properties, similar to those of Stratonovitch SDE based on a Brownian motion. In this section, we give the precise definition of the equation and state some basic properties of the solution, comparing it with that of the continuous Stratonovitch SDE

A canonical SDE on $\mathbf{R}^d$ is defined through an m-dimensional Lévy process $Z(t) = (Z^1(t), ..., Z^m(t)), 0 \leq t \leq T_0$ and $m + 1$ vector fields $V_0, V_1, ..., V_m$ on $\mathbf{R}^d$. It is denoted as follows.

$$\xi_t = \eta_0 + \int_{t_0}^t V_0(\xi_s)ds + \sum_{j=1}^m \int_{t_0}^t V_j(\xi_s) \diamond dZ^j(s). \tag{1}$$

Here, η_0 is an $\mathbf{R}^d$ valued random variable independent of $Z(t) - Z(t_0), t \geq t_0$ such that $E[|\eta_0|^p] < \infty$ holds for any $p > 1$. The integral of the right hand side $\int \cdots \diamond Z^j(s)$ is the *canonical stochatic integral* based on the Lévy process $Z^j(s)$. In order to define it precisely, the Lévy-Itô decomposition of the Lévy process $Z(t)$ is needed. For any given $\delta > 0$, $Z(t)$ is decomposed as

$$Z(t) = Z(0) + \sigma B(t) + \int_0^t \int_{|z| \leq \delta} z\tilde{N}(dsdz) + \int_0^t \int_{|z|>\delta} zN(dsdz) + b^\delta t. \tag{2}$$

Here, $B(t)$ is an m dimensional standard Brownian motion and σ is an $m \times m$-matrix. Further, $N(dsdz)$ is a Poisson random measure on $[0, T_0] \times \mathbf{R}^m$, independent of $B(t)$, such that its compensator is $\hat{N}(dtdz) = dt\mu(dz)$, where μ is the Lévy measure. Further, $\tilde{N} = N - \hat{N}$. In the following, we set $Z_c(t) = \sigma B(t) + b^\delta t$ and $Z_d(t) = Z(t) - Z_c(t)$.

Let $z = (z^1, ..., z^m) \in \mathbf{R}^m$ and consider the vector field $\sum_{j=1}^m z^j V_j$. Suppose that it is complete. We denote by $\phi_t^z(x), t \in \mathbf{R}$ the flow of diffeomorphism generated by it. Thus its value at $t = 1$ defines the map $x \to \phi_1^z(x) : \mathbf{R}^d \to \mathbf{R}^d$ of a diffeomorphism for any $z \in \mathbf{R}^m$.

Now the canonical stochastic differential equation is defined as follows.

$$\xi_t = \eta_0 + \int_{t_0}^t V_0(\xi_s)ds + \sum_{j=1}^m \left\{ \int_{t_0}^t V_j(\xi_s) \circ dZ_c^j(s) + \int_{t_0}^t V_j(\xi_{s-})dZ_d^j(s) \right\}$$
$$+ \sum_{t_0<s<t, \Delta Z(s)\neq 0} \left\{ \phi_1^{\Delta Z(s)}(\xi_{s-}) - \xi_{s-} - \sum_{j=1}^m V_j(\xi_{s-})\Delta Z^j(s) \right\}.$$

Here, $\int \cdots \circ dZ_c^j(s)$ denotes the Stratonovitch integral and $\int \cdots dZ_d^j(s)$ denotes the Itô integral.

Thereom 1. Suppose that the vector fields $V_j, j = 0, ..., m$ are of C^2-class and that V_j and their derivatives are all bounded. Then equation (1) has a unique global solution.

The trajectries $\{\xi_s, t_0 \leq s \leq T_0\}$ of the solution is right continuous and has left hand limits. The jumps of ξ_s occur only when the jumps of $Z(s)$ occur. If $Z(s)$ do not have jumps at s, then their trajectries move continuously like the solution of the Stratonovitch SDE based on $Z_c(s)$. If $Z(s)$ have jumps $\Delta Z(s)$ at time s, then the trajectries of the solution jump from points ξ_{s-} to $\phi_1^{\Delta Z(s)}(\xi_{s-})$. That is, they fly from points ξ_{s-} along the integral curve of the vector fields $\sum_j \Delta Z^j(s)V_j$ with infinite speed and land to $\phi_1^{\Delta Z(s)}(\xi_{s-})$. Then they repeat the similar movement inductively.

We will list up properties ot the solutions.

1) Stochastic flows of diffeomorphisms. (Fujiwara-Kunita [4])

Denote the solution starting from x at time t_0 as $\xi_{t_0,t}(x)$. Then we can take its nice modification with respect to parameters t_0, t and x so that the modification satisfies for almost all ω,

1) For any $t_0 < t$, the map $x \to \xi_{t_0,t}(x); \mathbf{R}^d \to \mathbf{R}^d$ is an onto diffeomorphism.

2) $\xi_{t_0,u} = \xi_{t,u} \circ \xi_{t_0,t}$ holds for any $t_0 < t < u$.

2) Coordinate free property. (Fujiwara [3])

Suppose that the vector fields $V_0, ..., V_m$ are tangent to a submanifold S of $\mathbf{R}^d$. If the solution of the canonical SDE starts from a point of S, then its trajectries are always on S. (Note that if we replace the canonical integral by Itô integral, then the solution could leave from S.) Further, the definition of the canonical SDE does not depend on the choice of the local coordinate. Hence its defintion can be extended to any manifold.

3) Wong-Zakai approximation. (Kunita [6])

We will approximate the trajectries of the Lévy process $Z(t)$ by a sequence of continuous polygonal trajectries $\{Z_n(t)\}$:

$$Z_n(t) = Z\Big(\frac{k}{n}\Big) + \frac{t - \frac{k}{n}}{n}\Big\{Z\Big(\frac{k+1}{n}\Big) - Z\Big(\frac{k}{n}\Big)\Big\}, \quad \text{if } \frac{k}{n} \leq t < \frac{k+1}{n}.$$

We will consider a sequence of stochastic ordinary differential equations.

$$\frac{d\varphi_n(t)}{dt} = V_0(\varphi_n(t)) + \sum_{j=1}^m V_j(\varphi_n(t))\dot{Z}_n^j(t), \qquad \varphi^{(n)}(t_0) = \eta_0,$$

where $\dot{Z}_n^j(t) = \frac{d}{dt}Z_n^j(t)$. The solution $\varphi_n(t)$ is a continuous stochastic process. For each t, the sequence $\{\varphi^{(n)}(t)\}$ converges weakly to the solution of the canonical SDE (1).

4) The support theory of Stroock-Varadhan type. (Kunita [8])

We assume that $b^0 = \lim_{\delta\to 0} b^\delta$ exists and is finite. Then the Lévy process $Z(t)$ is represented as

$$Z(t) = Z(0) + \sigma B(t) + \int_0^t \int_{|z|>0} zN(dsdz) + b^0 t.$$

Let $\mathbf{D}$ be the set of all maps $u : [0,T] \to \mathbf{R}^m$ such that $u(0) = 0$ and $u(t)$ are right continuous with left hand limits. We associate the Skorohod topology to $\mathbf{D}$. We denote by $\mathcal{U}$ the set of all $u \in \mathbf{D}$ which satisfies a) the number of jumps is at most finite, b) $\Delta u(s) = u(s) - u(s-) \in \mathrm{supp}(\mu)$, where $\mathrm{supp}(\mu)$ is the support of the Lévy measure μ, and c) Set $u_c(t) = u(t) - u_d(t)$, $u_d(t) = \sum_{s\le t} \Delta u(s)$. Then $u_c(t)$ is a piecewise smooth and continuous function with values in $\mathcal{R}$ (the image of the linear map $A = \sigma\sigma^T$). Then the closure of $\mathcal{U}$ with respect to the Skorohod topology is the support of the Lévy process $Z(t) - Z(0)$.

Now we set

$$\hat{V}_0 = V_0 + \sum_j b_j^0 V_j, \tag{3}$$

and consider an ordinary differential equation with jumps associated with $u(t) \in \mathcal{U}$:

$$\begin{aligned}\varphi(t) \quad &= \quad x + \int_{t_0}^t \hat{V}_0(\varphi(s))ds + \sum_{j=1}^m \int_{t_0}^t V_j(\varphi(s))\dot{u}_c^j(s)ds \\ &\qquad + \sum_{t_0\le s\le t} \{\phi_1^{\Delta u(s)}(\varphi(s-)) - \varphi(s-)\}.\end{aligned}$$

Let $\varphi_x^u(t)$ be its solution. We set $\Phi = \{\varphi_x^u; u \in \mathcal{U}, x \in S\}$, where S is the support of the distribution of η_0. It is a subset of $\mathbf{D}$. Then the support $\mathcal{S}$ of the canonical SDE (1) coincides with the closure of Φ with respect to the Skorohod topology.

Remark If the integral $\int_{0<|z|\le 1} |z|\mu(dz)$ is finite, then b^0 exists and is finite. Hence for any stable process with exponent $0 < \alpha < 1$, b^0 exists. On the other hand, if the Lévy measure μ is symmetric, b^0 exists and is equal to 0 even if $\int_{0<|z|\le 1} |z|\mu(dz)$ is infinite. Hence for any symmetric stable process, b^0 exists and is 0.

2. Existence and smoothness of the density of the distribution of the solution.

In the canonical SDE, if the driving process $Z(t)$ is a Brownian motion, then the SDE coincides with the Stratonovitch SDE. In this case, if the set of the vector fields $\{V_0, V_1, ..., V_m\}$ satisfies Hormander's hypoellipticity condition (Hormander Condition (H)), then the distribution of the solution has a C^∞ density function. The fact has been proved by Malliavin, Kusuoka-Stroock and others using the Malliavin calculus. In this paper, we discuss the existence and smoothness of the density function for the canonical SDE with jumps, under conditions which are slightly stronger than Hörmander condition (H).

Similar problems have been studied for various type of SDE with jumps after the fundamental work of Bismut. In Bismut [2], Bichteler-Gravereau-Jacod [1], Leandre [10] and Komatsu-Takeuchi [5], the case where the Lévy measure has a smooth density is studied. Recently, Picard [12] studied the case where the Lévy measure satisfies a condition similar to ours but the coefficients (vector fields) are nondegenerate.

In order that the distribution of the solution of the SDE driven by a Lévy process has a density function, the Lévy process should have the same property. Concerning this, we will introduce a nondegenerate Lévy process. Let $A = \sigma\sigma^T$. It is a covariance matrix of the Gaussian part $Z_c(1)$ of $Z(1)$. We will define the infinitesimal covariance of $Z_d(t)$. Set

$$v_{ij}(\rho) = \int_{|z|\leq\rho} z^i z^j \mu(dz), \quad v(\rho) = \int_{|z|\leq\rho} |z|^2 \mu(dz).$$

We assume that $v(\rho) > 0, \forall\rho > 0$ and we define nonnegative symmetric matrices B_ρ and B by

$$B_\rho = \left(\frac{v_{ij}(\rho)}{v(\rho)}\right), \quad B = \liminf_{\rho\to 0} B_\rho$$

Thus B is the greatest lower bound of the matrices B_ρ so that it satisfies, $(l, Bl) \leq \liminf_{\rho\to 0} (l, B_\rho l)\,, \quad \forall l \in \mathbf{R}^m$. If the matrix $A + B$ is invertible, the Lévy process is called *nondegenerate*.

Lemma 1. (Orey) (see Proposition 2.8.3 in Sato [13]) Suppose that the Lévy process is nondegenerate and that the Lévy measure μ satisfies the order condition

$$\liminf_{\rho\to 0} \frac{v(\rho)}{\rho^\alpha} > 0$$

for some $0 < \alpha < 2$. Then the distribution of $Z(t) - Z(0)$ has a C^∞ density function for any $t > 0$.

We will consider the vector fields $V_0, V_1, ..., V_m$ which define our SDE. Using these vector fields, we set

$$\Sigma_0 = \{V_1, ..., V_m\}, \quad \Sigma_j = \{[V_0, V], [V_i, V], i = 1, ..., m, V \in \Sigma_{j-1}\}, \quad j = 1, 2, ...$$

where $[\cdot,\cdot]$ is the Lie bracket. If $\dim \cup_{j\geq 0}\Sigma_j(x) = d$ is satisfied for any $x \in \mathbf{R}^d$, then $\{V_0, V_1, ..., V_m\}$ is said to satisfy *Hörmander condition (H)*.

Theorem 2. (Kunita-Oh [9]) Suppose that the canonical SDE satisfies the next two conditions.
(a) The Lévy process $Z(t)$ is nondegenerate and the Lévy measure satisfies the order condition for some $\alpha \in (0, 2)$.
(b) The vector fields $\{V_0, V_1, ..., V_m\}$ satisfy the Hörmander condition (H).
Then for any η_0 and $t_0 < t \leq T_0$, the distribution of the solution ξ_t has a density function.

Let us next consider the smoothness of the density function. For this, we have to look into the drift term of the SDE (1) in detail. Suppose that $b^0 = \lim_{\delta\to 0} b^\delta$ exists and is finite. Then b^0 can be regarded as the drift vector of the Lévy process $Z(t)$.

Let $\hat{V}_0$ be the vectof field of (3). We introduce another set of vector fields;

$$\hat{\Sigma}_0 = \Sigma_0, \quad \hat{\Sigma}_j = \{[\hat{V}_0, V] + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}[V_i,[V_j,V]], [V_i,V], i=1,...,m, V \in \hat{\Sigma}_{j-1}\}, \tag{4}$$

for $j = 1, 2, ...$ Then $\{\hat{V}_0, V_1, ..., V_m\}$ is said to satisfy the *uniform Hörmander condition (H)* if there exists a positive integer N_0, a nonnegative integer n_0 and a positive constant C such that

$$\sum_{j=0}^{N_0} \sum_{V \in \hat{\Sigma}_j} |l^T V(x)|^2 \geq \frac{C}{(1+|x|)^{n_0}} |l|^2, \quad \forall x \in \mathbf{R}^d \tag{5}$$

for any vector l. Our main result is stated as follows.

Theorem 3. Assume that b^0 exists and that $\{\hat{V}_0, V_1, ..., V_m\}$ satisfies the uniform Hörmander condition (H). Assume further that

$$|l^T V(x)|^2 \leq \frac{C'}{(1+|x|)^{n_0}} |l|^2 \tag{6}$$

holds for any vector l and $V \in \cup_{j=1}^{N_0} \hat{\Sigma}_j$. Then the law of ξ_t has a C^∞-density for any η_0 and $t_0 < t \leq T_0$.

If the vector b_0 does not exist, the statement of the result becomes more complicated. Given $\delta > 0$, we set

$$\hat{V}_0^\delta = V_0 + \sum_{i=1}^{m} b_i^\delta V_i. \tag{7}$$

and define

$$\hat{\Sigma}_0^\delta = \Sigma_0, \quad \hat{\Sigma}_j^\delta = \{[\hat{V}_0^\delta, V] + \frac{1}{2}\sum_{i,j=1}^{m} [V_i,[V_j,V]], [V_i,V], i=1,...,m, V \in \hat{\Sigma}_{j-1}^\delta\}, \tag{8}$$

for $j = 1, 2, ...$

Theorem 4. Assume that there exists a positive integer N_0, a nonnegative integer n_0 and a positive number δ_0 such that for any $0 < \delta < \delta_0$ the inequality

$$\sum_{j=0}^{N_0} \sum_{V \in \hat{\Sigma}_i^\delta} |l^T V(x)|^2 \geq \frac{C(\delta)}{(1+|x|)^{n_0}} |l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^m, \tag{9}$$

holds, where $C(\delta)$ are positive numbers with the property $\liminf_{\delta \to 0} C(\delta)/v(\delta)^2 = \infty$. Assume further that (6) holds for any vector l and $V \in \cup_{i=1}^{N_0} \hat{\Sigma}_i^\delta$, where C' may depend on δ. Then the distribution of ξ_t has a C^∞-density for any η_0 and $t_0 < t \leq T_0$.

For the proof of these theorems, we will develope the Malliavin calculus on the Wiener-Poisson space following the idea of Picard [12], who studied the Malliavin calculus on the Poisson space.

3. Malliavin calculus

Let $T=[0,T_0]$. Let Ω_1 be the set of all continuous maps $T\to\mathbf{R}^m$ such that $\omega_1(0)=0$. Let $\mathcal{F}_1$ be its σ-field. P_1 is a probability measure on $(\Omega_1,\mathcal{F}_1)$ such that $W(t):=\omega_1(t)$ is a standard Brownian motion. Let Ω_2 be the set of all integer valued measures ω_2 on $T\times\mathbf{R}^m$ such that $\omega_2(T\times\{0\})=0$. Let $\mathcal{F}_2$ be its σ-field. Let P_2 be a probability measure on $(\Omega_2,\mathcal{F}_2)$ such that $N(dtdz):=\omega_2(dtdz)$ is a Poisson random measure and its intensity measure is $\hat{N}(dtdz):=dt\mu(dz)$. On the product space $\Omega=\Omega_1\times\Omega_2, \mathcal{F}=\mathcal{F}_1\times\mathcal{F}_2$ we define a product measure $P=P_1\times P_2$. The triple $(\Omega,\mathcal{F},P)$ is called the Wiener-Poisson space.

Now let $F=F(\omega_1,\omega_2)$ be a random variable such that it is smooth in the sense of Malliavin with respect to the first variable ω_1. The Malliavin-Shigekawa derivative of F with respect to the first variable ω_1 is denoted by $\{D_tF, t\in T\}$. Further we set for $(t_1,...,t_j)\in T^j$ $D^j_{t_1,...,t_j}=D_{t_1}\cdots D_{t_j}$ and $\|D^jF\|=(\int_{T^j}|D^j_{t_1,...,t_j}F|^2dt_1\cdots dt_j)^{1/2}$. For $p\geq 1$ and positive integer k we define the norm $\|\ \|_{k,p}$ by $\|\ \|_{k,p}=(E[|F|^p]+\sum_{j=1}^k E[\|D^jF\|^p])^{1/p}$. $\mathbf{D}^{k,p}$ is the space of random variables F with finite norm.

Next for $u=(t,z)=(t,z^1,...,z^m)\in T\times\mathbf{R}^m$ we define the map $\varepsilon_u^-:\Omega_2\to\Omega_2$ by $\varepsilon_u^-\omega_2(A)=\omega_2(A\cap\{u\}^c)$. Further, we define the map $\varepsilon_u^+:\Omega_2\to\Omega_2$ by $\varepsilon_u^+\omega_2(A)=\omega_2(A\cap\{u\}^c)+1_A(u)$. Since $\omega_2(\{u\})=0$ holds for almost all ω_2, $\epsilon_u^-\omega=\omega$ holds a.s. P for any u. We define the difference operator $\tilde{D}_u$ by

$$\tilde{D}_uF=F\circ\varepsilon_u^+-F.$$

If it is differentiable with respect to $z=(z^1,...,z^m)$, we define $d\times m$-matrix $\partial\tilde{D}_{t,z}F$ by $(\partial_{z^1}\tilde{D}_{t,z}F,...,\partial_{z^m}\tilde{D}_{t,z}F)$.

Set $\mathbf{u}=(u_1,...,u_l)=((t_1,z_1),...,(t_l,z_l))=(\mathbf{t},\mathbf{z})$ and $|\mathbf{z}|=\max_{1\leq i\leq l}|z_i|$. For $\alpha=(\alpha_1,...,\alpha_l)$, $\alpha_i\in\{1,...m\}$ we set $\partial^\alpha_{\mathbf{z}}=\partial_{z_1^{\alpha_1}}\cdots\partial_{z_m^{\alpha_m}}$. We define $\varepsilon^+_{\mathbf{u}}=\varepsilon^+_{u_1}\circ\cdots\varepsilon^+_{u_l}$ and $\tilde{D}_{\mathbf{u}}=\tilde{D}_{u_1}\cdots\tilde{D}_{u_l}$. Suppose $\tilde{D}_{(\mathbf{t},\mathbf{z})}F\circ\varepsilon^+_{\mathbf{v}}$ is continuous with respect to $(t,\mathbf{z},\mathbf{v})$ and is differentiable with respect to z. We denote by $\tilde{\mathbf{D}}^{k,p}$ the set of all F such that $\partial^\alpha_{\mathbf{z}}\tilde{D}_{\mathbf{u}}F$ exists, $\partial^\alpha_{\mathbf{z}}\tilde{D}_{\mathbf{u}}F\circ\epsilon^+_{\mathbf{v}}$ is continuous and satisfies $\sup_{\mathbf{v}}E\left[\int_{T^{|\alpha|}}\sup_{|\mathbf{z}|\leq 1}|\partial^\alpha_{\mathbf{z}}\tilde{D}_{(\mathbf{t},\mathbf{z})}F\circ\varepsilon^+_{\mathbf{v}}|^pd\mathbf{t}\right]<\infty$.

Given a d-dimensional random variable F belonging to $\cap_{k,p}(\mathbf{D}_{k,p}\cap\tilde{\mathbf{D}}_{k,p})$, we define the Malliavin covariances R_ρ and R by

$$\begin{aligned} R_\rho &= \Big(\langle DF_i,DF_j\rangle_{L^2(T)}\Big)+\int_T(\partial\tilde{D}_{t,0}F)B_\rho(\partial\tilde{D}_{t,0}F)^Tdt,\\ R &= \Big(\langle DF_i,DF_j\rangle_{L^2(T)}\Big)+\int_T(\partial\tilde{D}_{t,0}F)B(\partial\tilde{D}_{t,0}F)^Tdt.\end{aligned}$$

Theorem 5. (Kunita-Oh [9]) 1) Suppose that the Malliavin covariance R is invertible a.s. Then the distribution of F has a density.
2) Suppose that there exists $\rho_0>0$ such that for any $p>1$ and n,

$$\sup_{u\in A(\rho_0)^n}\sup_{|l|=1}E[\sup_{\rho\leq\rho_0}(l^TR_\rho\circ\varepsilon^+_{\mathbf{u}}l)^{-p}]\leq C_{p,n}, \tag{10}$$

where $A(\rho_0)=\{(t,z);t\in T,|z|\leq\rho_0\}$. Then the distribution of F has a C^∞-density function.

We shall apply the above theorem to the solution of the canonical SDE. We shall only consider the case $t_0=0$ and $t=T_0$. Set $F=\xi_{T_0}$. Then

$F \in \cap_{k,p}(\mathbf{D}^{k,p} \cap \tilde{\mathbf{D}}^{k,p})$. We shall compute the Malliavin covariance. We can show similarly as the case of diffusion that $D_t F$ is represented as

$$D_t F = \Big(\nabla \xi_{t,T_0}(\xi_{t-})\Big(\sum_i V_i(\xi_{t-})\sigma_{i,j}\Big), j = 1, ..., m\Big), \quad a.e. \ \ dtdP.$$

Here ξ_t is the solution of equation (1) and $\nabla \xi_{t,T_0}$ is the Jacobian matrix of the diffeomorphism ξ_{t,T_0}. It is invertible. On the other hand, $F \circ \varepsilon^+_{(t,z)}$ is written as $F \circ \varepsilon^+_{(t,z)} = \xi_{t,T_0} \circ \phi^z_1 \circ \xi_{t-}$. By the mean value theorem, we get

$$\tilde{D}_{t,z} F = \nabla \xi_{t,T_0}(\phi^z_\theta \circ \xi_{t-})\Big(\sum_i z^i V_i(\phi^z_\theta \circ \xi_{t-})\Big), \quad a.e. \ \ dtdP.$$

Therefore, $\partial \tilde{D}_{t,0} F = \nabla \xi_{t,T_0}(\xi_{t-}) V(\xi_{t-})$, where, $V(x) = (V_1(x), ..., V_m(x))$. Consequently the Malliavin covariance R is written by

$$R = \int_T \nabla \xi_{t,T_0}(\xi_{t-}) V(\xi_{t-})(A + B) V(\xi_{t-})^T \nabla \xi_{t,T_0}(\xi_{t-})^T dt, \quad a.s. \ \ P \tag{11}$$

Instead of the above, it is convenient to consider the *modified Malliavin covariance* $\hat{R} = (\nabla \xi_{0,T_0})^{-1} R (\nabla \xi_{0,T_0})^{-1,T}$. It is written as

$$\hat{R} = \int_0^{T_0} (\nabla \xi_{0,t})^{-1} V(\xi_t) V(\xi_t)^T (\nabla \xi_{0,t})^{-1,T} dt. \tag{12}$$

We have

Theorem 6. (1) The distribution of $F = \xi_{T_0}$ has a density if the modified Malliavin covariance $\hat{R}$ is invertible a.s. P.
(2) The distribution of $F = \xi_{T_0}$ has a C^∞-density if the modified Malliavin covariance $\hat{R}$ is invertible and the inverse satisfies

$$\sup_{u \in A(\rho_0)^n} \sup_{|l|=1} E((l^T \hat{R} \circ \varepsilon^+_{\mathbf{u}} l)^{-p}) < \infty, \quad \forall p > 1 \tag{13}$$

for some $\rho_0 > 0$.

Theorem 2 can be verified using the first part of the above theorem. Given a vector field V, we shall consider a right continuous semimartingale $Y_V(t) = l^T (\nabla \xi_{0,t})^{-1} V(\xi_t)$. Then the modified Malliavin covariance is represented by $l^T \hat{R} l = \sum_{V \in \Sigma_0} \int_0^{T_0} |Y_V(t)|^2 dt$.

Lemma 2. (Kunita-Oh [9]) $Y_V(t)$ is written as

$$\begin{aligned} Y_V(t) = Y_V(0) &+ \int_0^t Y_V^{(0)}(s-) ds + \sum_{j=1}^m \sigma_{\cdot j} \int Y_V^{(j)}(s-) dW^j_s \\ &+ \int_0^t \int_{|z| \le \delta} Y_V^{(1)}(s-, z)|z| d\tilde{N} + \int_0^t \int_{|z| > \delta} Y_V^{(1)}(s-, z)|z| dN. \end{aligned} \tag{14}$$

where,

$$\begin{aligned} Y_V^{(0)}(t) = {} & l^T (\nabla \xi_{0,t})^{-1} \{[\hat{V}_0^\delta, V] + \frac{1}{2} \sum_{i,j=1}^m a_{ij} [V_i, [V_j, V]]\}(\xi_t) \\ &+ \int_0^t \int_{|z| < \delta} (\nabla \xi_{0,t})^{-1} \{(\nabla \phi^z_1)^{-1} V(\phi^z_1 \circ \xi_s) - V(\xi_s) - \sum z^i [V_i, V](\xi_s)\} d\hat{N} \end{aligned}$$

$$\begin{aligned} Y_V^{(j)}(t) &= l^T(\nabla\xi_{0,t})^{-1}[V_j,V](\xi_t), \\ Y_V^{(1)}(t,z) &= l^T(\nabla\xi_{0,t})^{-1}\Psi_1(z)V(\xi_t), \end{aligned}$$

and $\Psi_1(z) = \Phi_1(z)/|z|$.

Proof of Theorem 2. It is convenient to consider the following family of vector fields

$$\Sigma_0' = \Sigma_0, \quad \Sigma_j' = \{[\hat{V}_0, V] + \frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]], [V_i,V], i=1,...,m, V \in \hat{\Sigma}_{j-1}\},$$

for $j = 1, 2, ...$ Then it holds $\cup_{j=0}^\infty \Sigma_j' = \cup_{j=0}^\infty \Sigma_j$. Hence if Hörmander's Condition (H) is satisfied, then $\cup_j \Sigma_j'(x) = \mathbf{R}^d$ holds for any $x \in \mathbf{R}^d$.

Now suppose that for a vector l, $l^T\hat{R}l = 0$ holds a.s. Then, we have $l^T(\nabla\xi_{0,t})^{-1}V(\xi_t) = 0$ for $0 < \forall t < T_0$ a.s. for any $V \in \Sigma_0'$. We apply Lemma 2 for $Y_V(t) = l^T(\nabla\xi_{0,t})^{-1}V(\xi_t)$. Then each term of the right hand side of (12) is 0. Therefore, for any $V \in \Sigma_0'$, we have $Y_V^{(0)}(t) = 0$, $\sum_j \sigma_{ij}Y_V^{(j)}(t) = 0$ for $i = 1, ..., m$ and $Y_V^{(1)}(t,z) = 0$ for $z \in \mathrm{Supp}(\mu)$. Consider the second and the third equality. The second implies

$$\sum_{i,j} a_{ij}Y_{[V_i,V]}(t)Y_{[V_j,V]} = 0.$$

Note that $\partial_{z^k}Y_V^{(1)}(t,z)\Big|_{z=0} = Y_{[V_k,V]}(t)$. Then the third one implies

$$\sum_{i,j} b_{ij}Y_{[V_i,V]}(t)Y_{[V_j,V]} = 0.$$

Since $A + B$ is invertible, we get $Y_{[V_i,V]}(t) = 0$ for any $i = 1, ..., m$.

We shall next consider $Y_V^{(0)}(t)$. Using the above equality, it is written simply as

$$Y_V^{(0)}(t) = l^T(\nabla\xi_{0,t})^{-1}\{[V_0,V] + \frac{1}{2}\sum_{i,j} a_{ij}[V_i,[V_j,V]]\}(\xi_t).$$

Since it is 0, we have obtained the equality $Y_V(t) = 0$ for any $V \in \Sigma_1'$.

Repeating this argument, we have $Y_V(t) = 0$ for any $V \in \Sigma_j'$, $0 < \forall t < T_0$. Now it holds $\dim \cup_j \Sigma_j'(x) = d$ by Hörmander condition (H). Therefore we get $l = 0$. Hence $\hat{R}$ and R are invertible a.s.

4. Smooth densities of distributions of solutions to canonical SDE

4.1. Another density theorem.

We shall introduce a *modified uniform Hörmander condition*. Given $\delta > 0$, we define a linear transformation Ψ_0^δ of vector fields by

$$\begin{aligned} \Psi_0^\delta V &= [\hat{V}_0^\delta, V] + \frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]] \\ &\quad + \int_{0<|z|\le\delta}\left((\phi_1^{-z})_*V - V - \sum_{i=1}^m [V_i,V]z^i\right)\mu(dz), \end{aligned}$$

where $(\phi_1^{-z})_*$ is the differential of the diffeomorphism ϕ_1^{-z}. We may consider $\Psi_0^\delta V$ as a modification of the vector field $[\hat{V}_0^\delta, V]$. We define

$$\Gamma_0^\delta = \Sigma_0, \quad \Gamma_j^\delta = \{\Psi^\delta V, [V_i, V], i = 1, ..., m, V \in \Gamma_{j-1}^\delta\}, \quad j = 1, 2, ...$$

These can be regarded as a modification of $\hat{\Sigma}_j^\delta$ of (8). $\{V_0, V_1, ..., V_m\}$ is said to satisfy the *modified uniform Hörmander condition (H) for* δ if there exists a positive integer N_0, a nonnegative integer n_0 and a positive constant C_3 such that

$$\sum_{j=0}^{N_0} \sum_{V \in \Gamma_j^\delta} |l^T V(x)|^2 \geq \frac{C_3}{(1+|x|)^{n_0}} |l|^2, \quad \forall x \in \mathbf{R}^d \tag{15}$$

for any vector l.

Theorems 3-4 stated in Section 2 can be obtained easily from the following theorem.

Theorem 7 Assume that $\{V_0, V_1, ..., V_m\}$ satisfy the modified uniform Hörmander condition (H) for some δ. Assume further that (6) holds for any vector l and $V \in \cup_{j=1}^{N_0} \Gamma_j^\delta$. Then the law of ξ_{T_0} has a C^∞-density.

4.2. Estimate of Norris' type.

The proof of Theorem 7 is very long. Here we give the outline of the proof of the above theorem. The complete proof will be discussed elsewhere.

We want to prove, under the modified uniform Hörmander condition (H), that for any $p > 1$ and n, there exists $C_{p,n} > 0, \varepsilon_0 > 0$ such that

$$\sup_{\mathbf{u} \in A(\rho_0)^n} \sup_{|l|=1} P(l^T \hat{R} \circ \varepsilon_{\mathbf{u}}^+ l < \varepsilon) < C_{p,n} \varepsilon^p \tag{16}$$

holds for any $0 < \varepsilon < \varepsilon_0$, where $\hat{R}$ is the modified Malliavin covariance (12). Indeed, if the the above holds valid, then $\sup_{\mathbf{u} \in A(\rho_0)^n} \sup_{|l|=1} E[(l^T \hat{R} \circ \varepsilon_{\mathbf{u}}^+ l)^{-p}] < \infty$ and the assertion of the theorem follows. In order to prove (16), we need an estimate similar to the one obtained by Kusuoka-Stroock and Norris in case of diffusion.

Let $b^\gamma(t), e^\gamma(t) = (e_1^\gamma(t), ..., e_m^\gamma(t)), f^\gamma(t) = (f_1^\gamma(t), ..., f_m^\gamma(t)), g^\gamma(t, z), h^\gamma(t, z)$ be left continuous predictable processes, continuous with respect to the parameter $z \in \mathbf{R}^m, \gamma \in \Gamma$, where Γ is a compact space. We consider a semimartingale

$$\begin{aligned} Y_t^\gamma = y^\gamma + \int_0^t a^\gamma(s) ds + \sum_i \int_0^t f_i^\gamma(s) dW_s^i \\ + \int_0^t \int_{|z| \leq \delta} g^\gamma(s, z) d\tilde{N} + \int_0^t \int_{|z| > \delta} g^\gamma(s, z) dN \end{aligned} \tag{17}$$

where $a^\gamma(t)$ is also a semimartingale represented by

$$\begin{aligned} a^\gamma(t) = \alpha^\gamma + \int_0^t b^\gamma(s) ds + \sum_i \int_0^t e_i^\gamma(s) dW_s^i \\ + \int_0^t \int_{|z| \leq \delta} h^\gamma(s, z) d\tilde{N} + \int_0^t \int_{|z| > \delta} h^\gamma(s, z) dN. \end{aligned}$$

We set $u^\gamma(t)^2 = \int_{\mathbf{R}^m} g^\gamma(t,z)^2\mu(dz)$, $v^\gamma(t)^2 = \int_{\mathbf{R}^m} h^\gamma(t,z)^2\mu(dz)$ and

$$\theta^\gamma(t) = |a^\gamma(t)|^2 + |e^\gamma(t)|^2 + |f^\gamma(t)|^2 + |u^\gamma(t)|^2 + |b^\gamma(t)|^2 + |v^\gamma(t)|^2 + \sup_{|z|\geq\delta} |h^\gamma(t,z)|^2.$$

We assume that

$$c_p = E\Big[\sup_{t,\gamma} \theta^\gamma(t)^p\Big] < \infty \tag{18}$$

holds for any $p > 1$. We set $\hat{g}^\gamma(t,z) = \frac{g^\gamma(t,z)}{|z|}$.

Lemma 3. (c.f. Komatsu-Takeuchi [5]) Let Y_t^γ be a semimartingale represented as above. Let $\beta > 0$ be a number such that $\alpha(1+\beta) < 2$ and let $q, r > 0$ be such that $q > 4r$ and $r > \frac{1}{2-\alpha(1+\beta)}$. Then for any $C_0 > 0$ and $p > 1$, there exists $\varepsilon_0 > 0$ and $C_p > 0$ such that the inequality

$$P\Big\{ \int_\Gamma \Big(\int_0^{T_0} |Y_{t-}^\gamma|^2 \wedge \varepsilon^{2r} dt\Big)\pi(d\gamma) < \varepsilon^q, \quad \int_\Gamma \Big(\int_0^{T_0} (a^\gamma(t)^2 + |f^\gamma(t)|^2)dt \\ + \int_0^{T_0} \int_{\mathbf{R}^m} \hat{g}^\gamma(t,z)^2 \wedge \varepsilon^{-2\beta r} dt \hat{\mu}_{\varepsilon^{r(1+\beta)}}(dz)\Big)\pi(d\gamma) > C\varepsilon \Big\} \leq C_p\varepsilon^p \tag{19}$$

holds for all $0 < \varepsilon < \varepsilon_0$, $C > C_0$ and probability measures π on Γ. Here,

$$\hat{\mu}_\rho(dz) = \frac{1}{v(\rho)}|z|^2 1_{[0,\rho]}(|z|)\mu(dz). \tag{20}$$

4.3. Outline of the proof of Theorem 7.

We want to prove Theorem 7 by applying the second part of Theorem 6. It is convenient to introduce the following notations. We set $S = \hat{\mathbf{R}}^m \cup \mathbf{R}^m \cup \{\Delta\}$. Elements of $\hat{\mathbf{R}}^m$ and $\mathbf{R}^m$ are denoted by $y = (y^1, ..., y^m)$ and $z = (z^1, ..., z^m)$, respectively. Associated with a vector field V, we define a stochastic process $Y_V(t,u)$ with parameter $u \in S$ by

$$\begin{aligned} Y_V(t,\Delta) &= l^T(\nabla\xi_{0,t})^{-1}\Psi_0^\delta V(\xi_t), \\ Y_V(t,y) &= \sum_{i=1}^m l^T(\nabla\xi_{0,t})^{-1}[V_i, V](\xi_t)\frac{y^i}{|y|}, \\ Y_V(t,z) &= l^T(\nabla\xi_{0,t})^{-1}\frac{\Phi_1(z)}{|z|}V(\xi_t). \end{aligned}$$

Let $W(dsdy)$ be a Gaussian orthogonal random measure on $[0,T] \times \hat{\mathbf{R}}^m$ such that $E[W(dsdy)] = 0$ and $\sigma W_t = \int_0^t \int_{\hat{\mathbf{R}}^m} yW(dsdy)$. Then the intensity measure $E(W(dsdy)^2) = dsw(dy)$ satisfies $(\int_{\hat{\mathbf{R}}^m} y^i y^j w(dy)) = A$. We set $\hat{w}(dy) = |y|^2 w(dy)$. Then Lemma 2 implies

$$\begin{aligned} Y_V(t) &= Y_V(0) + \int_0^t Y_V(s-,\Delta)ds + \int_0^t \int_{\hat{\mathbf{R}}^m} Y_V(s-,y)|y|dW \\ &\quad + \int_0^t \int_{|z|\leq\delta} Y_V(s-,z)|z|d\tilde{N} + \int_0^t \int_{|z|>\delta} Y_V(s-,z)|z|dN. \end{aligned} \tag{21}$$

Set

$$E_0 = \left\{ \sum_{V\in\Sigma_0} \int_0^{T_0} |Y_V(t-)|^2 dt < \varepsilon \right\},$$

$$E_1 = \left\{ \sum_{V\in\Sigma_0} \int_0^{T_0} \int_S |Y_V(t-,u)|^2 \wedge \varepsilon^{-2\beta r/q} dt \nu_{\varepsilon^{(1+\beta)r/q}}(du) < \varepsilon^{1/q} \right\}.$$

Here ν_ϵ is a measure on S such that ν_ϵ equals $\hat{w}$ on $\hat{\mathbf{R}}^m$, equals $\hat{\mu}_\epsilon$ on $\mathbf{R}^m$ and equals δ_Δ on Δ. By applying Lemma 3 we can show

$$P(E_0 \cap E_1) \le mC_p\epsilon^p. \tag{22}$$

We will continue the above argument inductively. Let $j \ge 1$. We will define a family of j-th step semimartingales with spatial parameter associated with a given vector field V. We set $\Psi(\Delta)V = \Psi_0^\delta V$, $\Psi(y)V = \sum_k [V_k, V]y^j/|y|$ and $\Psi(z)V = \Phi_1(z)V/|z|$. Define for $u_j, ..., u_1 \in S$, $\Psi(u_j, ..., u_1)V = \Psi(u_j) \circ \cdots \circ \Psi(u_1)V$ and

$$Y_V(t, u_j, ..., u_1) = l^T(\nabla\xi_{0,t})^{-1}\Psi(u_j, ..., u_1)V(\xi_t).$$

We will apply Lemma 3 again by setting $\pi(d\lambda) = \nu_{\varepsilon^{q(j)}}(du_j)\cdots\nu_{\varepsilon^{q(1)}}(du_1)$, where $q(j) = (1+\beta)rq^{-j}, j = 1, 2, ...$ Set for $0 < \varepsilon < \varepsilon_0$,

$$E_j = \left\{ \sum_{V\in\Sigma_0} \int_0^{T_0} \int |Y_V(t-, u_j, ..., u_1)|^2 \wedge \varepsilon^{-2\beta rq^{-j}} dt \nu_{\varepsilon^{q(j)}}(du_j)\cdots\nu_{\varepsilon^{q(1)}}(du_1) < \varepsilon^{q^{-j}} \right\}.$$

Then it holds

$$P(E_j \cap E_{j+1}^c) \le 2^{j+1}mC_p\varepsilon^p, \quad j = 1, 2, ... \tag{23}$$

for all $0 < \varepsilon < \varepsilon_0$.

Now, we have the relation

$$E_0 \subset (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{N_0-1} \cap E_{N_0}^c) \cup G,$$

where

$$G = E_0 \cap E_1 \cap \cdots \cap E_{N_0},$$

and N_0 is a positive integer appearing in (15). We want to get the estimate of $P(E_0)$. We have already obtained the following estimate.

$$\sup_{l;|l|=1} P(\cup_{j=0}^{N_0-1}(E_j \cap E_{j+1}^c)) \le 2^{N_0+1}mN_0C_p\varepsilon^p. \tag{24}$$

In the following, we will get the estimate

$$\sup_{l;|l|=1} P(G) < C'_p\varepsilon^p, \tag{25}$$

for all $0 < \varepsilon < \varepsilon_0$. If this is verified, then (24) and (25) imply

$$P(\sum_{V\in\Sigma_0} |\int_0^{T_0} Y_V(t-)|^2 dt < \epsilon) \le C_p\epsilon^p.$$

Hence we get (16) in the case where $\mathbf{u} = 0$ and $n = 0$.

In order to prove (25), we will sum up all random variables which define sets $E_0, E_1, ..., E_{N_0}$. Note that it depends on ε. We denote it by K_ε. Then it is written as

$$K_\varepsilon = \int_0^{T_0} L_\varepsilon(l^T(\nabla\xi_{t-})^{-1}, \xi_{t-})dt,$$

where

$$L_\varepsilon(l,x) = \sum_{V\in\Sigma_0} \Big\{ |l^T V(x)|^2 + \sum_{j=1}^{N_0} \int\cdots\int \Big(|l^T\Psi(u_j,...,u_1)V(x)|^2 \wedge \varepsilon^{-2\beta r q^{-j}} \Big) \nu_{\varepsilon^{q(j)}}(du_j)\cdots\nu_{\varepsilon^{q(1)}}(du_1) \Big\}.$$

We can obtain the lower bound of $L_\varepsilon(l,x)$, making use of the modified Hörmander condition (H).

Lemma 4. Assume the modified uniform Hörmander condition (H) for some $\delta > 0$. Then there exists $0 < \epsilon_0 < 1$ such that the inequality

$$L_\varepsilon(l,x) \geq \frac{\hat{\lambda}_1^{N_0} C_3}{4} \frac{|l|^2}{(1+|x|)^{n_0}}$$

holds for any $0 < \varepsilon < \varepsilon_0$. Here, λ_1 is the minimum eigen value of the matrix $A+B$ and $\hat{\lambda}_1 = \min\{\lambda_1, 1\}$. Further, N_0 is a positive integer and C_3 is a positive constant appearing in (15).

The proof is omitted.

The above lemma leads to

$$K_\varepsilon \geq \frac{\hat{\lambda}_1^{N_0} C_3}{4} \int_0^{T_0} \frac{|l^T(\nabla\xi_{t-})^{-1}|^2}{(1+|\xi_{t-}|)^{n_0}} dt,$$

if $\varepsilon < \varepsilon_0$. Now, if $\omega \in G$, we have the inequality $K_\varepsilon(\omega) < \sum_{j=0}^{N_0} \varepsilon^{q^{-j}} < (N_0+1)\varepsilon^{q^{-N_0}}$ if $\varepsilon^{1/q} < 1$. Therefore, we have $G \subset \{K_\varepsilon < (N_0+1)\varepsilon^{q^{-N_0}}\}$. Further, for any l with $|l| = 1$, we have

$$\left(\int_0^{T_0} \frac{|l^T(\nabla\xi_{t-})^{-1}|^2}{(1+|\xi_{t-}|)^{n_0}} dt \right)^{-1} \leq \frac{1}{T_0^2} \int_0^{T_0} |\nabla\xi_{t-}|^2 (1+|\xi_{t-}|)^{n_0} dt,$$

by using Jensen's inequality. Therefore,

$$G \subset \left\{ \int_0^{T_0} |\nabla\xi_{t-}|^2 (1+|\xi_{t-}|)^{n_0} dt > \frac{\hat{\lambda}_1^{N_0} C_3 T_0^2}{4(N_0+1)\varepsilon^{q^{-N_0}}} \right\}.$$

Then we get $P(G) \leq C_p' \varepsilon^p$ by Chebyschev's inequality. We have thus obtained the estimate (25) for all $0 < \varepsilon < \varepsilon_0$.

So far we proved

$$\sup_{|l|=1} P(l^T \hat{R} l < \varepsilon) < C_p \varepsilon^p.$$

Then we can easily reduce the stronger assertion (16) to the above.

References

[1] Bichteler, K., Gravereaux, J.B., Jacod,J., Malliavin calculus for processes with jumps. Stochastic Monographs, vol.2, London: Gordon and Breach 1987

[2] Bismut, J.M., Calcul des variations stochastique et processus de sauts. Z. W. 63, 147-235(1983)

[3] Fujiwara, T., Stochastic differential equations of jump type on manifolds and Lévy flows, J. Math., Kyoto Univ. 31(1991), 99-119.

[4] Fujiwara, T., Kunita, H., Canonical SDE's based on semimartingales with apatial parameters, Part I, Stochastic flows of diffeomorphisms, Kyushu J. Math. 53(1999), 265-300.

[5] Komatsu, T., Tekeuchi T., On the smoothness of pdf of solutions to SDE of jumps type, preprint.

[6] Kunita,H., Some problems concerning Lévy processes on Lie groups, Stochastic analysis, Proceedings of symposia in pure mathematics, 57(1995), 323-342, AMS

[7] Kunita, H., Stochastic differential equation with jumps and stochastic flows of diffeomorphisms, Itô's stochastic calculus and probability theory, ed by Ikeda-Watanabe-Fukushima-Kunita, 197-211, Springer, 1996

[8] Kunita, H., Canonical stochastic differential equations based on Lrvy processes and their supports, Stochastic Dymnamics,ed. by H. Crauel, M. Gundlach, 283-304, Springer 1999.

[9] Kunita, H., Oh, J., Malliavin valculus on the Wiener-Poisson space and its applications to canonical SDE with jumps, in preparation

[10] Leandre, R., Regularites de processus de sauts degeneres (II). Ann. Inst. Henri Poincare Prob. Stat. 24, 209-236(1988)

[11] Marcus, S.I., Modelling and approximation of stochastic differential equations driven by semimartingales, Stochastics 4(1981), 223-245.

[12] Picard, J., On the existence of smooth densities for jump processes, PTRF 105, 481-511(1996)

[13] Sato, K., *Lévy processes and infinitely divisible distributions*, Cambridge, 1999